Licenciatura em Gestão

Operational Research Chapter 6

2017-2018

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Integer Linear Programming arises when in the context of Linear Programming the decision variables only make sense if they have integer values, that is if the assumption of divisibility of LP doesn't fit to the problem in hands.

Examples
- to decide on: - the number of machines to purchase; - the number of people to select for a job; - the type of projects to invest on;
- "yes-or-no" decisions (for example, to invest or not invest in some project, ...) .

Def. 1:
- A ILP is **a pure ILP** if **all** the decision variables are required to have integer values;
- A **mixed ILP** is a ILP where only **some** variables are required to have integer values.

Def. 2:
The **linear relaxation of an ILP** is the LP obtained by dropping the constraints on the integer values for the variables (the integrality constraints).

Integer Linear Programming (ILP)

Parameters of the model are:

c_j (j=1,2,...,n) coefficient of variable j at the Objective Function;

b_i (i=1,2,...,m) right-hand-side of constraint i;

a_{ij} (i=1,2,...,m; j=1,2,...,n) technical coefficient.

Let x_j be the level of the activity j (j=1,...,n) - units of product j that should be produced, and by Z the total performance (revenue).

The ILP model and the respective linear relaxation are:

Integer Linear Problem (ILP)

$$Z^* = Max\ Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.}\begin{cases} \sum_{j=1}^{n} a_{ij}x_j \le b_i & i = 1, 2, \dots, m \\ x_j \ge 0\ and\ integer & j = 1, 2, \dots, n \end{cases}$$

Linear Relaxation (LR)

$$Z_R^* = Max\ Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.}\begin{cases} \sum_{j=1}^{n} a_{ij}x_j \le b_i & i = 1, 2, \dots, m \\ x_j \ge 0 & j = 1, 2, \dots, n \end{cases}$$

Property: Let Z^* be the optimum value of a maximization ILP and Z_R^* the optimum value of its linear relaxation, then: $Z^* \le Z_R^*$.

43. *TBA Airlines* is a small air company, specialized in regional flights. The management is considering an expansion and it has the possibility to buy small or medium size airplanes. Find the best strategy, knowing that at the moment <u>no more than two small airplanes</u> can be bought and that $100 *millions* are available to invest. Consider also the values in the following table:

	small airplane	medium size airplane
annual profit per airplane	$1 million	$5 millions
cost per airplane	$5 millions	$50 millions

1) Solve by the graphical method;
2) Solve by solver/excel.

Integer Linear Programming (ILP)

- **Graphical method**

    1. solve graphically the Linear Relaxation (LR) of the problem. If the variables restricted to integer values have integer values, then the optimal solution of the ILP is found. Otherwise, identify the feasible region of the ILP within the LR feasible region.
    2. consider the objective function and the feasible region of the ILP and determine the optimal solution, as usual.

- **Resolution by the solver/excel**

    1. in an excel spreadsheet write the problem to solve and prepare the resolution by solver as if the problem is a linear programming one.
    2. fill the solver parameters window as usual.
    3. add the constraints on the integrality of variables: the cells of the variables are integer.
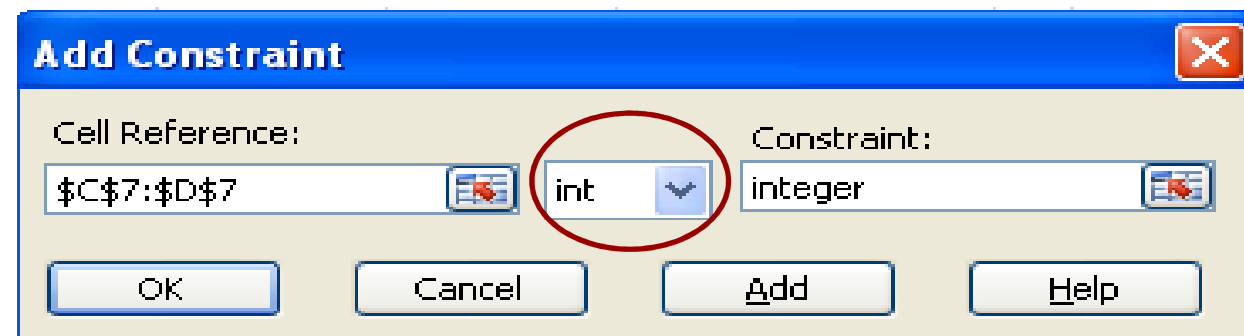
Integer Linear Programming (ILP)

Formulations with binary variables

<u>Binary variables</u> assume only one of the values: zero or one.

Binary Integer Programming (BIP) Problems are ILP with binary variables. If all variables are restricted to binary values it is a Pure BIP, if some variables are binary it is a Mixed BIP.

Binary variables are useful to model a wide range of situations:

• Mutually exclusive alternatives (such that if a decision is chosen others cannot be)

• Contingent decisions (decisions that depend upon other decisions)

• Alternative constraints (either - or constraints)

• Incorporate fixed and variable costs of products

• Complementary products

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Integer Linear Programming (ILP)

Prototype Example – *CALIFORNIA MANUFACTURING COMPANY* (HL[1], §11.1, pg. 465)

The CALIFORNIA MANUFACTURING COMPANY is considering expansion by building a new factory in either Los Angeles (LA) or San Francisco (SF), or perhaps even in both cities. It also is considering building at most one new warehouse, but the choice of a location is restricted to a city where a new factory is being built. The *net present value* (total profitability considering the time value of money) of each of these alternatives is shown in the second column of the table below. The right most column gives the capital required (already included in the net present value) for the respective investments, where the total capital available is $10 *million.* The objective is to find the feasible combination of alternatives that maximizes the total net present value*.

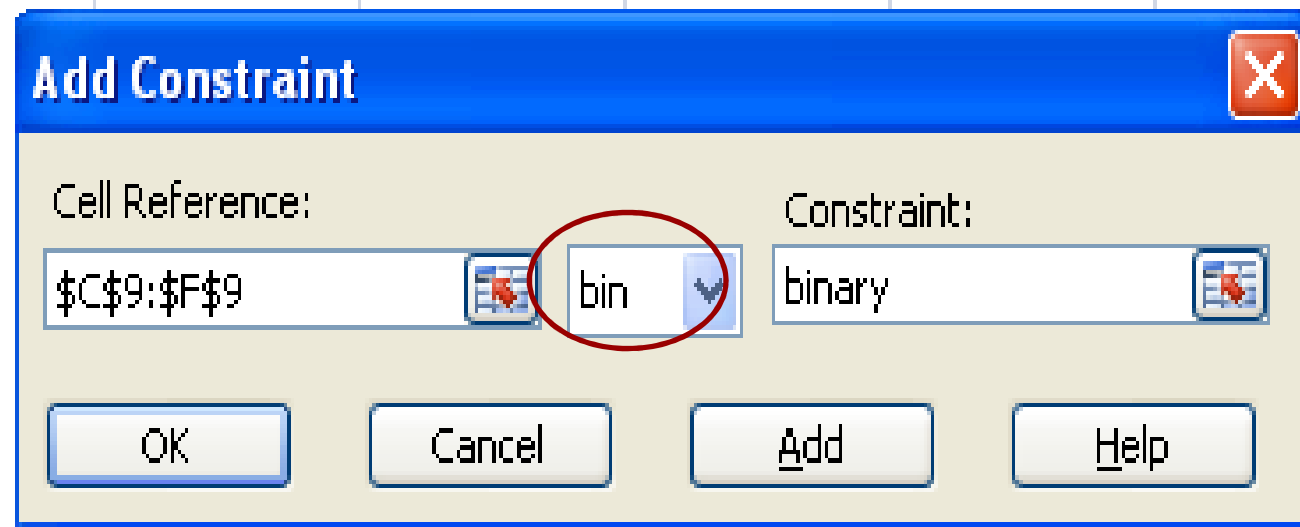| Build a	Net Present Value	Capital Requirement
Factory in LA	$9 million	$6 million
Factory in SF	$5 million	$3 million
Warehouse in LA	$6 million	$5 million
Warehouse in SF	$4 million	$2 million

1 Hillier and Lieberman, "Introduction to Operations Research", 9th ed., McGraw-Hill 2010.

Integer Linear Programming (ILP)

Solving Prototype Example by solver/excel

Proceed as usual, except that new constraints on the values of the variables must be added.

In the solver parameters window **Add constraint**

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Example: **Mutually exclusive decisions**

Consider the **Prototype Example 1** – Wyndor Glass, Co (chapter 1)

x_1 $-$ no. batches of P1 produced per week (P1=8-foot glass door with aluminum framing)
x_2 $-$ no. batches of P2 produced per week (P2=4×6 foot double-hung wood framed window)
Z $-$ total profit per week (in m.u.) from producing these two products

Linear Programming (LP) Model

$$Max\ Z = 3x_1 + 5x_2$$

$$s.t. \begin{cases} x_1 & \leq 4 \\ & 2x_2 \leq 12 \\ 3x_1 + & 2x_2 \leq 18 \\ x_1, x_2 \geq 0 \end{cases}$$

- **Suppose that** the company cannot produce both products simultaneously.

Integer Linear Programming (ILP)

Example: Mutually exclusive decisions (cont.)

Define binary variables y_j

$$y_j = \begin{cases} 1 \; if \; product \; j \; is \; produced \\ \\ 0 \; otherwise \end{cases} \quad (j=1,2)$$

where $j=1$ respect to doors and $j=2$ to the windows.

- In the LP model of Wyndor Glass Co, include the constraints:

$$y_1 + y_2 \leq 1$$

$$x_j \leq My_j \; (j=1,2)$$

$$y_j \in \{0,1\} \; (j=1,2)$$

where M is a sufficiently big positive number (for example, $M=1000$).

Integer Linear Programming (ILP)

Example: Either-Or Constraints

Consider again the Prototype Example 1 – Wyndor Glass, Co

- Suppose that Plant 3 may be replaced by other plant (Plant 4) if more profitable.
  Plant 4 has 52 hours per week available. In this plant the production of each batch
  of doors uses 3 h while each batch of windows requires 8 h.

Define binary variables y_i

$$y_i = \begin{cases} 1 \; if \; constraint \; i \; is \; active \\ \\ 0 \; otherwise \end{cases} \quad (i = 3,4)$$

Integer Linear Programming (ILP)

Example: Either-Or Constraints (cont.)

In the LP model of Wyndor Glass Co,

- substitute the 3rd functional constraint by $\quad 3x_1 + 2x_2 \leq 18 + M(1 - y_3)$ (Plant 3)

- insert the 4th functional constraint $\qquad 3x_1 + 8x_2 \leq 52 + M(1 - y_4)$ (Plant 4)

- insert also the constraints

$$y_3 + y_4 = 1$$

$$y_3, y_4 \in \{0,1\}$$

where M is a sufficiently big positive number (for example, $M=1000$)

Integer Linear Programming (ILP)

Example: Fixed-charge (setup) cost

Consider again the Prototype Example 1 – Wyndor Glass, Co

- Suppose that setup costs of 7 m.u. and 13 m.u. should be associated to the production of doors and windows, respectively.

Define binary variables y_j

$$y_j = \begin{cases} 1 \ if \ j \ is \ produced \\ \\ 0 \ otherwise \end{cases} \qquad (j = 1,2)$$

where $j=1$ respect to doors and $j=2$ to the windows.

Integer Linear Programming (ILP)

Example: Fixed-charge (setup) cost (cont.)

In the LP model of Wyndor Glass Co,

- change the objective function to $Max\ Z = 3x_1 + 5x_2 - 7y_1 - 13y_2$

- add the constraints:

$$x_j \leq My_j \quad (j=1,2)$$

$$y_j \in \{0,1\} \quad (j=1,2)$$

where M is a sufficiently big positive number (for example, M=1000).